Teton dam failure, June 5, 1976, Teton Canyon, Idaho.

IS DISPERSION IMPORTANT IN FLOOD ROUTING?

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14 November 2023

ABSTRACT. A convection-diffusion-dispersion equation of flood flows (**Ferrick and others, 1984)** is used as the basis for the development of a dimensionless convection-diffusion-dispersion equation. This equation shows that its three coefficients are functions only of the Froude and Vedernikov numbers, recognized as the two conceptual pillars of unsteady open-channel hydraulics. The computer program **ONLINEDISPERSIVITY** is used to establish the order of magnitude of dispersivity to guide actual routing computations.

1. INTRODUCTION

Flood routing is the calculation of the movement of a flood wave in space and time along a stream or channel. The governing equations are the equations of water continuity and motion of open-channel flow, the so-called Saint-Venant (1871) equations (Chow, 1959; **Ponce, 2014a**). The numerical solution of these equations leads to a *mixed kinematic-dynamic wave*. In hydraulic engineering practice, this wave is commonly referred to as the "dynamic wave" (Fread, 1985).

The realization that the role of inertia is very often minimal led **Hayami (1951)** to simplify the flood routing problem by combining the set of two governing equations into *one* equation, with space x and time t as independent variables and discharge Q as the dependent variable. This equation is referred to as the *convection-diffusion* equation. It is a partial differential equation of second order, describing convection, of first order, and diffusion, of second order. This approach to flood routing has been referred to as Hayami's *diffusion analogy* (**Ponce, 2014b**).

Ferrick and others (1984) added another term to the convection-diffusion equation, effectively creating a third term, which they characterized as *dispersion*. Thus arose the convection-diffusion-dispersion equation of flood routing.. Ferrick's work was further enhanced by **Ponce (2020)**, who expressed the convection-diffusion-dispersion equation in dimensionless form. Ponce expressed the coefficients of this equation in terms of *only* the Froude and Vedernikov numbers, thereby rationalizing the entire subject of unsteady open-channel flow (**Ponce, 2023**).

This article analyzes the significance of the dispersion term in both the theory and practice of hydraulic engineering. The online calculator **ONLINEDISPERSIVITY** is used to calculate relevant parameters to streamline the flood routing analysis.

A word of caution.

The term d*ispersion* was used by **Ferrick and others (1984)** to denote the third-order term in the governing differential equation of free-surface flow. The term has also been used in Computational Hydraulics to describe the *third-order term* in flood routing applications for both the analytical equation and its numerical analog; e.g., *numerical dispersion* (**Ponce, 2014b)**. Actually, the term "dispersion" may mean different things to different people. For instance, in fluid mechanics, dispersion generally refers to the *spreading of mass* from higher to lower concentrations. In this article, we use the term **dispersion** in the mode of Ferric and others, to refer to the third-order spread of momentum in open-channel flow.

2. CONVECTION-DIFFUSION-DISPERSION EQUATION

Table 1, Equation 1, shows the convection-diffusion-dispersion equation. The convection coefficient is the Seddon, or kinematic wave celerity (**Seddon, 1900**; **Ponce, 2014b**). The diffusion coefficient is the Hayami diffusivity (**Hayami, 1951**; **Ponce, 2014b**). The dispersion coefficient is the Ferrick diffusivity (**Ferrick and others, 1984**; **Ponce, 2020**).

Table 1. Elements of the convection-diffusion-dispersion equation.		
Equation	$Q_t + c\, Q_x = \nu\, Q_{xx} + \eta\, Q_{xxx}$	(1)
Convection coefficient	$c = \left(1 + \dfrac{\mathbf{V}}{\mathbf{F}} \right) u_o$	(2)
Diffusion coefficient	$\nu = \dfrac{L_o}{2}\, u_o \left(1 - \mathbf{V}^2 \right)$	(3)
Dispersion coefficient	$\eta = \left(\dfrac{L_o}{2} \right)^2 u_o \left(1 - \mathbf{V}^2 \right) \mathbf{F}^2$	(4)
Symbol definition. Q = discharge; A = flow area; x = space; t = time; $\mathbf{V}$ = Vedernikov number = $(\beta - 1)\, \mathbf{F}$; β = exponent of the discharge-area rating $Q = \alpha A^{\beta}$; $\mathbf{F}$ = Froude number = $u_o / (g\, y_o)^{1/2}$; u_o = mean velocity; y_o = flow depth; g = gravitational acceleration; S_o = channel bottom slope; L_o = reference channel length = y_o / S_o.		

3. DIMENSIONLESS CONVECTION-DIFFUSION-DISPERSION EQUATION

Table 2, Equation 5, shows the dimensionless convection-diffusion-dispersion of flood waves (**Ponce, 2020**). To accomplish the nondimensionalization, we used the reference channel length L_o, which is the distance along the channel in which the channel drops a head equal to its flow depth (Table 2, bottom) (**Lighthill and Whitham, 1955**; **Ponce and Simons, 1977**). *All three* dimensionless coefficients are shown to be functions *only* of the Froude and Vedernikov numbers. Therefore, we conclude that these two numbers effectively constitute the pillars of unsteady open-channel flow. Together, they describe and characterize wave motion.

It is observed that the dimensionless convection coefficient c' (Eq. 6), which may also be referred to as *dimensionless kinematic wave celerity*, is in fact the exponent of the discharge-flow area rating: $\beta = 1 + (\mathbf{V}/\mathbf{F})$. Thus, β may be regarded as perhaps the most significant parameter in the entire field of unsteady open-channel flow (**Ponce, 2023**).

Table 2. Elements of the dimensionless convection-diffusion-dispersion equation.		
Equation	$Q_{t'} + c' Q_{x'} = \nu' Q_{x'x'} + \eta' Q_{x'x'x'}$	(5)
Dimensionless convection coefficient	$c' = 1 + \dfrac{\mathbf{V}}{\mathbf{F}}$	(6)
Dimensionless diffusion coefficient	$\nu' = \dfrac{1}{2} (1 - \mathbf{V}^2)$	(7)
Dimensionless dispersion coefficient	$\eta' = \dfrac{1}{4} (1 - \mathbf{V}^2) \mathbf{F}^2$	(8)
Symbol definition. x' = dimensionless space = x / L_o; t' = dimensionless time = $t (u_o / L_o)$; $\mathbf{F}$ = Froude number = $u_o / (g y_o)^{1/2}$; u_o = mean velocity; y_o = flow depth; g = gravitational acceleration; $\mathbf{V}$ = Vedernikov number = $(\beta - 1)\ \mathbf{F}$; β = exponent of the discharge-area rating $Q = \alpha A^{\beta}$.		

4. ANALYSIS

Table 3 shows the results of script **ONLINEDISPERSIVITY**. We varied mean velocity u_o (Col. 2), flow depth y_o (Col. 3), and channel bottom slope S_o (Col. 4) as shown. The focus was on unit-width discharge ($q_o = u_o y_o$) and channel bottom slope S_o (Col. 4), since these two variables are strongly related to the diffusion (Eq. 3) and dispersion (Eq. 4) coefficients. The value of β, the exponent of the discharge-area rating, was fixed at $\beta = 1.5$ (Col. 5) because in practice it varies within a relatively narrow range.

Specific observations regarding the coefficients of diffusion and dispersion are the following:

- Diffusion (Col. 9) increases strongly with an increase in unit-width discharge, i.e., with a simultaneous increase in both u_o and y_o (Cols. 2 and 3).

- Diffusion (Col. 9) increases strongly with a decrease in channel slope (Col. 4).

- Dispersion (Col. 10) increases very strongly with an increase in unit-width discharge, i.e., with a simultaneous increase in both u_o and y_o (Cols. 2 and 3).
- Dispersion (Col. 10) increases very strongly with a decrease in channel slope (Col. 4).

Table 3. Results of script ONLINEDISPERSIVITY.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
Line	u_o (m/s)	y_o (m)	S_o (m/m)	β	F	V	c (m/s)	ν (m^2/s)	η (m^3/s)	c'	ν'	η'
1	1	1	0.01	1.5	0.32	0.16	1.5	4.97	248.34	1.5	0.49	0.025
2	1	1	0.001	1.5	0.32	0.16	1.5	49.7	24834.	1.5	0.49	0.025
3	1	1	0.0001	1.5	0.32	0.16	1.5	497.	2483475.	1.5	0.49	0.025
4	2	2	0.01	1.5	0.45	0.225	3.0	38.69	3870.	1.5	0.47	0.048
5	2	2	0.001	1.5	0.45	0.225	3.0	386.9	386964.	1.5	0.47	0.048
6	2	2	0.0001	1.5	0.45	0.225	3.0	3869.	38696497.	1.5	0.47	0.048
7	4	4	0.01	1.5	0.64	0.32	6.0	292.94	58585.	1.5	0.45	0.092
8	4	4	0.001	1.5	0.64	0.32	6.0	2929.4	5858924.	1.5	0.45	0.092
9	4	4	0.0001	1.5	0.64	0.32	6.0	29294.	585892404.	1.5	0.45	0.092

The magnitude of the dispersion coefficients (Col. 10), particularly for the mild slopes (Lines 3, 6, and 9), indicates that the order of magnitude of the dispersion effect could be comparable to that of the diffusion effect. This remains to be confirmed in actual routing computations.

5. SUMMARY

A convection-diffusion-dispersion equation of flood flows (**Ferrick and others, 1984**) is used as the basis for the development of a dimensionless convection-diffusion-dispersion equation. This equation shows that its three coefficients are functions *only* of the Froude and Vedernikov numbers, recognized as the two conceptual pillars of unsteady open-channel hydraulics. The computer program **ONLINEDISPERSIVITY** is used to establish the order of magnitude of dispersivity to guide actual routing computations.

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